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Discontinuous Galerkin method for a class of elliptic multi-scale problems

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SUMMARY

In this paper we develop a discontinuous Galerkin (DG) method for solving a class of second-order elliptic multi-scale problems. The main ingredient of this method is to use a non-polynomial multi-scale approximation space in the DG method to capture the multi-scale solutions using coarse meshes without resolving the fine-scale structure of the solution. We perform analysis on the approximation, stability and error estimates, and provide numerical results to demonstrate the proposed method. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we design a discontinuous Galerkin (DG) method based on non-polynomial basis functions [1] for solving a class of multi-scale second-order elliptic partial differential equations

$$-\nabla \cdot (a^{\varepsilon}(x)\nabla u) = f(x) \quad \text{in } \Omega \tag{1}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega$$

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where Ω is a rectangular domain and the coefficient $a^{\varepsilon}(x)$ is an oscillatory function involving a small scale ε, for example it could be $a^{\varepsilon}(x)=a(x, x/\varepsilon)$; however the force function $f(x)$ does not involve this small scale. A standard DG method [2, 3] using piecewise polynomial approximation spaces can be used to solve (1); however it would require a very fine mesh to resolve the small scale ε in the solution, which might be too costly or impossible on today's computers.

We will explore the usage of non-polynomial multi-scale approximation spaces in the DG method, consisting of special basis functions constructed from the partial differential equations (PDEs) to capture the micro-scale structure information of the solutions so that the solutions of multi-scale PDEs can be well approximated even on coarse meshes. Similar approaches have been used by Babuška *et al.* [4–6] and by Hou and Wu [7] for continuous finite element methods, in which both theoretical proofs and numerical experiments were provided to show that more accurate results can be obtained by using the multi-scale approximation spaces instead of piecewise polynomial spaces. We extend the methodology to the DG methods, providing analysis on the approximation, stability and error estimates, and numerical results to demonstrate the proposed method. The DG method, comparing with the continuous finite element methods, has the advantage of more flexibility on non-conforming meshes with hanging nodes, easier h-p adaptivity, and more importantly easier patching of the local multi-scale approximation spaces in different cells, especially for multi-dimensional problems, since no continuity across cell interfaces is required.

This paper is organized as follows. In Section 2, we give a brief review of the DG method for elliptic PDEs. In Section 3, the new multi-scale approximation spaces are introduced and some approximation results are given. In Section 4, stability and approximation properties are proven for the DG method based on the multi-scale approximation spaces. Then, error estimates are obtained in Section 5. In Section 6, a numerical example in one space dimension is provided to demonstrate the performance of the DG method. Concluding remarks and plans for future work are given in Section 7.

2. THE DISCONTINUOUS GALERKIN METHOD

In this section we give a brief review of some DG finite element methods for elliptic problems [2, 8–10]. For the sake of simplicity, we only consider the second-order elliptic problem in 1-D:

$$-(a(x)u_x)_x = f(x), \quad 0 \leqslant x \leqslant 1 \tag{2}$$

where $a(x)>0$. We first rewrite the problem as a first-order system

$$-(aw)_x = f, \quad w - u_x = 0 \tag{3}$$

Assuming $I_j=(x_{j-1/2}, x_{j+1/2})$, $j=1,\ldots,N$, is a partition of [0, 1], we multiply both equations by test functions v and τ, respectively, and integrate over the cell I_j, followed by an integration by parts to obtain the weak formulation:

$$\int_{I_j} a w v_x \,\mathrm{d}x - a_{j+1/2} w_{j+1/2} v_{j+1/2} + a_{j-1/2} w_{j-1/2} v_{j-1/2} = \int_{I_j} f v \,\mathrm{d}x \tag{4}$$

$$\int_{I_j} w\tau \,\mathrm{d}x + \int_{I_j} u \tau_x \,\mathrm{d}x - u_{j+1/2}\tau_{j+1/2} + u_{j-1/2}\tau_{j-1/2} = 0 \tag{5}$$

where, e.g. $u_{j+1/2}=u(x_{j+1/2})$.

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The general formulation of the DG method for the elliptic problem (2) is: Find $U, W \in V_h$ such that

$$\int_{I_j} aWv_x\,\mathrm{d}x - a_{j+1/2}\hat{W}_{j+1/2}v^-_{j+1/2} + a_{j-1/2}\hat{W}_{j-1/2}v^+_{j-1/2} = \int_{I_j} fv\,\mathrm{d}x \tag{6}$$

$$\int_{I_j} W\tau\,\mathrm{d}x + \int_{I_j} U\tau_x\,\mathrm{d}x - \hat{U}_{j+1/2}\tau^-_{j+1/2} + \hat{U}_{j-1/2}\tau^+_{j-1/2} = 0 \tag{7}$$

for all test functions $v, \tau \in V_h$. Here V_h is a finite dimensional finite element space containing functions that are discontinuous across cell interfaces, and, e.g. $v^-_{j+1/2} = v(x^-_{j+1/2})$. For traditional DG methods, these functions are piecewise polynomials. In this paper, we consider multi-scale approximation spaces. Equations (6)–(7) are called the *flux formulation*. We can also eliminate the auxiliary variable W from the *flux formulation* (6)–(7) to obtain a typical finite element formulation, which is called the *primal formulation*.

There are many choices for the fluxes $\hat{U}$ and $\hat{W}$. Different choices will give us different DG methods. We will just give two examples below.

First, the *flux formulation* of the LDG method [9–11] is: (6)–(7) with the following choice of fluxes:

$$\hat{U}_{j+1/2} = U^-_{j+1/2}, \quad \hat{W}_{j+1/2} = W^+_{j+1/2} + [U]_{j+1/2} \quad \text{or}$$

$$\hat{U}_{j+1/2} = U^+_{j+1/2}, \quad \hat{W}_{j+1/2} = W^-_{j+1/2} + [U]_{j+1/2}$$

where $[U] = U^+ - V^-$.

Secondly, the *primal formulation* of the Babuška–Zlámal method [12] is: Find $U \in V_h$ such that

$$\sum_{j=1}^{N} \int_{I_j} a(x)U_x v_x\,\mathrm{d}x + \sum_{j=0}^{N} \left(a_{j+1/2}\frac{\eta_{j+1/2}}{\Delta x}[U]_{j+1/2}[v]_{j+1/2} \right) = \sum_{j=1}^{N} \int_{I_j} fv\,\mathrm{d}x$$

for all test functions $v \in V_h$, where $\eta_{j+1/2}$ is a positive constant. There is a theoretical requirement for the lower bound $\eta_0 = \inf_j \eta_{j+1/2}$ for maintaining stability and rates of convergence of this method: $\eta_0 \approx \Delta x^{-2k}$ if $\dim V_h|_{I_j} = k+1$.

For the multi-scale problem (1), a direct numerical solution of this problem by the DG method based on standard piecewise polynomial spaces is difficult when ε is very small, even with modern supercomputers. The reason is that piecewise polynomial spaces do not approximate well the solution of (1) when the mesh is not refined enough. We need to use a very fine mesh in the computation to get good numerical results, hence the computational cost will be huge.

The core idea of the DG method in this paper is to construct the finite element basis functions that can capture the small scale information. In this DG method, the basis functions are constructed from the elliptic differential operator, hence they are adapted to the local properties of the differential operator in each cell. This idea of the construction of basis functions was first used by Babuška *et al.* to 1-D problems [5, 6] and to a special class of 2-D problems [4], and also by Hou and Wu to some 2-D elliptic problems with rapidly oscillating coefficients [7], in the context of continuous finite elements. For continuous finite element methods, there exists difficulty in enforcing continuity of basis functions at the 2-D cell interfaces. However, for the DG methods, continuity at the element interfaces is not needed, thus eliminating this difficulty.

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3. MULTI-SCALE APPROXIMATION SPACES

The multi-scale approximation spaces are constructed as below:

$$S^1 = \{\phi : \nabla \cdot (a^\varepsilon(x)\nabla\phi)|_K = 0\} \tag{8}$$

and

$$S^k = \{\phi : \nabla \cdot (a^\varepsilon(x)\nabla\phi)|_K \in P^{k-2}(K)\} \quad \text{for } k\geqslant 2 \tag{9}$$

where K denotes the cell in the space discretization. If we define $P^{-1}(K)=\{0\}$, (8) can also be denoted by (9) for $k=1$.

First consider the 1-D elliptic problem

$$-(a^\varepsilon(x)u_x)_x = f(x), \quad 0\leqslant x\leqslant 1 \tag{10}$$

with the boundary condition

$$u(0)=u(1)=0 \tag{11}$$

where

$$0<\alpha\leqslant a^\varepsilon(x)\leqslant\beta<+\infty \tag{12}$$

The multi-scale approximation space (9) for this problem is explicitly given

$$S^k = \left\{ v : v|_{I_j} \in \text{span}\left\{ 1, \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi, \int_{x_j}^x \frac{\xi - x_j}{a^\varepsilon(\xi)}\,\mathrm{d}\xi, \ldots, \int_{x_j}^x \frac{(\xi - x_j)^{k-1}}{a^\varepsilon(\xi)}\,\mathrm{d}\xi \right\}\right\} \tag{13}$$

where $x_j = \frac{1}{2}(x_{j-1/2} + x_{j+1/2})$. This multi-scale approximation space (13) has a very good approximation property for the solution of problem (10). This is supported by the following lemmas.

Lemma 3.1
Let $u(x)$ be the exact solution of (10). There exists some $v(x)\in S^k (k\geqslant 1)$ such that for all j:

$$|u(x) - v(x)|\leqslant C(\alpha,\beta)|f|_{H^{k-1}(I_j)}(\Delta x_j)^{k+1/2} \quad \forall x\in I_j \tag{14}$$

where $C(\alpha,\beta)$ is a constant depending on α, β and independent of Δx_j and ε.

Proof
The proof is inspired by the fundamental theorem of calculus:

$$\begin{aligned} u(x) &= u(x_j) + \int_{x_j}^x u'(y)\,\mathrm{d}y = u(x_j) + \int_{x_j}^x a^\varepsilon(y)u'(y)\frac{1}{a^\varepsilon(y)}\,\mathrm{d}y \\ &= u(x_j) + \int_{x_j}^x a^\varepsilon(y)u'(y)\,\mathrm{d}\left(\int_x^y \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right) \end{aligned}$$

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$$= u(x_j) + a^{\varepsilon}(x_j)u'(x_j)\int_{x_j}^{x} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi - \int_{x_j}^{x}\left(\int_{x}^{y} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}(a^{\varepsilon}(y)u'(y))$$

$$= u(x_j) + a^{\varepsilon}(x_j)u'(x_j)\int_{x_j}^{x} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi - \int_{x_j}^{x}\left(\int_{y}^{x} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right) f(y)\,\mathrm{d}y$$

where in the last equality we have used the differential equation (10). When $k=1$, let $v = u(x_j) + a^{\varepsilon}(x_j)u'(x_j)\int_{x_j}^{x}(1/a^{\varepsilon}(\xi))\,\mathrm{d}\xi$. It holds that $v \in S^1$ and moreover

$$|u(x)-v(x)| = \left|-\int_{x_j}^{x}\left(\int_{y}^{x} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right) f(y)\,\mathrm{d}y\right| \leqslant \left|\int_{x_j}^{x}\left(\int_{y}^{x} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right)^2 \mathrm{d}y\right|^{1/2} \left|\int_{x_j}^{x} f^2(y)\,\mathrm{d}y\right|^{1/2}$$

$$\leqslant \frac{1}{\alpha}(\Delta x_j)^{3/2}\|f\|_{L^2(I_j)} \tag{15}$$

for all $x \in I_j$.

When $k \geqslant 2$, we can find some $p \in P^{k-2}(I_j)$ such that

$$\|f - p\|_{L^2(I_j)} \leqslant C|f|_{H^{k-1}(I_j)}(\Delta x_j)^{k-1}$$

Here and below C is a generic constant independent of the functions and meshes sizes. In particular, C is independent of the small parameter ε. Let $v = u(x_j) + a^{\varepsilon}(x_j)u'(x_j)\int_{x_j}^{x}(1/a^{\varepsilon}(\xi))\,\mathrm{d}\xi - \int_{x_j}^{x}(\int_{y}^{x}(1/a^{\varepsilon}(\xi))\,\mathrm{d}\xi)p(y)\,\mathrm{d}y$. We can easily verify that $v \in S^k$ since

$$\int_{x_j}^{x}\left(\int_{y}^{x} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right) p(y)\,\mathrm{d}y = \int_{x_j}^{x}\left(\int_{x_j}^{\xi} p(y)\,\mathrm{d}y\right)\frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi$$

and $\int_{x_j}^{\xi} p(y)\,\mathrm{d}y \in P^{k-1}(I_j)$. Then we can obtain

$$|u(x) - v(x)| = \left|-\int_{x_j}^{x}\left(\int_{x}^{y} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right)(f(y) - p(y))\,\mathrm{d}y\right|$$

$$\leqslant \left|\int_{x_j}^{x}\left(\int_{x}^{y} \frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\right)^2 \mathrm{d}y\right|^{1/2} \left|\int_{x_j}^{x}(f(y) - p(y))^2\,\mathrm{d}y\right|^{1/2}$$

$$\leqslant \frac{1}{\alpha}(\Delta x_j)^{3/2}\|f - p\|_{L^2(I_j)} \leqslant C(\alpha,\beta)(\Delta x_j)^{k+1/2}|f|_{H^{k-1}(I_j)} \tag{16}$$

for all $x \in I_j$.

Combining (15) and (16) will finish the proof. □

Next, we estimate the approximation rate in the L^2 norm.

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Lemma 3.2
Let $u(x)$ be the exact solution of (10) and P_h be the L^2 projection operator into the space S^k. There exists a constant $C(\alpha, \beta)$ such that

$$\|u - P_h u\|_{L^2(0,1)} \leqslant C(\alpha, \beta)|f|_{H^{k-1}(0,1)}(\Delta x)^{k+1} \tag{17}$$

Proof
We choose the same v as that in Lemma 3.1. Squaring both sides of (14) and then integrating in the cell I_j, we obtain, for any j,

$$\|u - v\|^2_{L^2(I_j)} \leqslant C(\alpha, \beta)|f|^2_{H^{k-1}(I_j)}(\Delta x_j)^{2k+2}$$

Therefore,

$$\|u - P_h u\|^2_{L^2(0,1)} \leqslant \|u - v\|^2_{L^2(0,1)} = \sum_j \|u - v\|^2_{L^2(I_j)}$$

$$\leqslant C(\alpha, \beta) \sum_j |f|^2_{H^{k-1}(I_j)}(\Delta x_j)^{2k+2} \leqslant C(\alpha, \beta)|f|^2_{H^{k-1}(0,1)}(\Delta x)^{2k+2}$$

Taking square roots on both sides finishes the proof. □

We now show a numerical example in Table I for the approximation to the solution of the elliptic problem (10). We choose

$$f = x, \quad a = \frac{1}{2 + x + \sin(2\pi x/\varepsilon)} \tag{18}$$

with different choices of ε. We can see that we obtain the optimal order of the approximation rate (equal to the dimension of the local approximation space) when using the approximation spaces S^k $(k = 1, 2)$, starting from a very coarse mesh with the mesh size Δx larger than ε, verifying (17). For comparison, we also list in Table I the approximation results using the regular DG space of piecewise polynomials. We notice that we do not observe the expected order of convergence until the mesh is refined enough relative to ε, which is consistent with the approximation theory for such regular DG space of piecewise polynomials.

We also show that the usual inverse inequality holds for the multi-scale approximation spaces.

Lemma 3.3 (*Inverse Inequality*)
For any $\phi \in S^k$, we have the following inverse inequality:

$$\|\phi\|_{L^\infty(I_j)} \leqslant C(\Delta x_j)^{-1/2}\|\phi\|_{L^2(I_j)} \tag{19}$$

for some constant C.

Proof
Only the proof for $k = 1$ will be given here. $\{1, \int_{x_j}^{x}(1/a^\varepsilon(\xi))\,d\xi\}$ is a set of basis functions of the local space $S^1|_{I_j}$. We can transfer it to an L^2 orthogonal basis $\{b_0, b_1\}$ with

$$b_0 = 1, \quad b_1 = \frac{1}{\Delta x_j}\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,d\xi\right)dy$$

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Table I. L^2-errors of the approximation to the solution of problem (10) with the choice of (18), based on the multi-scale spaces and regular piecewise polynomials. Uniform mesh with N cells.

	$\varepsilon=0.1$				$\varepsilon=0.01$			
	S^1 space		P^1 space		S^1 space		P^1 space	
N	L^2-error	Order	L^2-error	Order	L^2-error	Order	L^2-error	Order
10	5.58E − 04		1.46E − 03		5.92E − 04		6.67E − 04	
20	1.49E − 04	1.91	2.55E − 04	2.52	1.48E − 04	2.00	1.74E − 04	1.94
40	3.71E − 05	2.01	1.54E − 04	0.73	3.73E − 05	1.99	2.21E − 04	−0.34
80	9.36E − 06	1.99	3.96E − 05	1.96	9.18E − 06	2.02	1.62E − 04	0.45
160	2.35E − 06	1.99	9.97E − 06	1.99	2.26E − 06	2.02	7.78E − 05	1.06
320	5.87E − 07	2.00	2.50E − 06	2.00	5.85E − 07	1.95	2.30E − 05	1.76
	S^2 space		P^2 space		S^2 space		P^2 space	
N	L^2-error	Order	L^2-error	Order	L^2-error	Order	L^2-error	Order
10	7.82E − 06		4.61E − 04		7.94E − 06		3.51E − 05	
20	1.00E − 06	2.97	2.03E − 04	1.18	9.97E − 07	2.99	1.04E − 05	1.75
40	1.26E − 07	2.99	2.02E − 05	3.33	1.27E − 07	2.97	2.18E − 04	−4.39
80	1.60E − 08	2.98	2.58E − 06	2.97	1.51E − 08	3.07	1.21E − 04	0.85
160	2.00E − 09	3.00	3.24E − 07	2.99	1.87E − 09	3.01	2.68E − 05	2.17
320	2.51E − 10	2.99	4.06E − 08	3.00	2.44E − 10	2.97	3.84E − 06	2.80

First we prove that for each i, b_i satisfy the inverse inequality (19). The proof is trivial for $i=0$. For $i=1$, we have

$$|b_1| = \frac{1}{\Delta x_j}\left|\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y\right| \leqslant \frac{1}{\Delta x_j}\int_{I_j}\left|\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right|\mathrm{d}y \leqslant \frac{1}{\alpha\Delta x_j}\int_{I_j}|x-y|\,\mathrm{d}y \leqslant \frac{\Delta x_j}{\alpha}$$

that is

$$\|b_1\|_{L^\infty(I_j)} \leqslant \frac{\Delta x_j}{\alpha} \tag{20}$$

From symmetry, we have

$$\int_{I_j}\left(\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y\right)\mathrm{d}x = 0$$

Hence there exists some $x_0 \in I_j$ such that

$$\int_{I_j}\left(\int_y^{x_0} \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y = 0$$

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We claim that for $x>x_0$, $\int_{I_j}(\int_y^x (1/a^\varepsilon(\xi))\,\mathrm{d}\xi)\,\mathrm{d}y>0$, since

$$\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y=\int_{I_j}\left(\int_{x_0}^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi+\int_y^{x_0} \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y=\int_{I_j}\left(\int_{x_0}^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y>0$$

Similarly, for $x<x_0$, we have $\int_{I_j}(\int_y^x (1/a^\varepsilon(\xi))\,\mathrm{d}\xi)\,\mathrm{d}y<0$.

Without the loss of generality, we may assume $x_0\leqslant x_j$. We then have

$$\begin{aligned}
(\Delta x_j)^2\|b_1\|^2_{L^2(I_j)}&=\int_{I_j}\left(\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y\right)^2\mathrm{d}x\\
&\geqslant\int_{x_0}^{x_j+\Delta x_j/2}\left(\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y\right)^2\mathrm{d}x\\
&\geqslant\frac{1}{x_j+\dfrac{\Delta x_j}{2}-x_0}\left(\int_{x_0}^{x_j+\Delta x_j/2}\left(\int_{I_j}\left(\int_y^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y\right)\mathrm{d}x\right)^2\\
&\geqslant\frac{1}{\Delta x_j}\left(\int_{x_0}^{x_j+\Delta x_j/2}\left(\int_{I_j}\left(\int_{x_0}^x \frac{1}{a^\varepsilon(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}y\right)\mathrm{d}x\right)^2\\
&\geqslant\frac{1}{\Delta x_j}\left(\int_{x_0}^{x_j+\Delta x_j/2}\frac{\Delta x_j}{\beta}(x-x_0)\,\mathrm{d}x\right)^2\geqslant\frac{(\Delta x_j)^5}{64\beta^2}
\end{aligned}$$

That is, we have

$$\|b_1\|_{L^2(I_j)}\geqslant\frac{(\Delta x_j)^{3/2}}{8\beta} \tag{21}$$

Combining the two inequalities (20) and (21), we obtain

$$\|b_1\|_{L^\infty(I_j)}\leqslant C(\Delta x_j)^{-1/2}\|b_1\|_{L^2(I_j)}$$

For any $\phi\in S^1$, we can write $\phi=c_0b_0+c_1b_1$, and we have

$$\begin{aligned}
\|\phi\|^2_{L^\infty(I_j)}&\leqslant 2c_0^2\|b_0\|^2_{L^\infty(I_j)}+2c_1^2\|b_1\|^2_{L^\infty(I_j)}\leqslant C(\Delta x_j)^{-1/2}(c_0^2\|b_0\|^2_{L^2(I_j)}+c_1^2\|b_1\|^2_{L^2(I_j)})\\
&=C(\Delta x_j)^{-1/2}\|\phi\|^2_{L^2(I_j)}
\end{aligned}$$

Now the proof is complete for $k=1$. □

We now consider a special 2-D elliptic multi-scale problem

$$-(a^\varepsilon(x)u_x)_x-(b^\varepsilon(y)u_y)_y=f(x,y),\quad (x,y)\in\Omega \tag{22}$$

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with the boundary condition

$$u(x,y)=0, \quad (x,y)\in\partial\Omega$$

where

$$0<\alpha\leqslant a^{\varepsilon}(x), \quad b^{\varepsilon}(y)\leqslant\beta<+\infty \tag{23}$$

We propose a 2-D multi-scale approximation space for this 2-D elliptic problem to be

$$\begin{aligned}S_2^k=\Bigg\{v: v|_K\in\operatorname{span}\Bigg\{&1, \int_{x_K}^{x}\frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi, \int_{y_K}^{y}\frac{1}{b^{\varepsilon}(\eta)}\,\mathrm{d}\eta, \int_{x_K}^{x}\frac{\xi-x_K}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi,\\ &\int_{x_K}^{x}\frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\int_{y_K}^{y}\frac{1}{b^{\varepsilon}(\eta)}\,\mathrm{d}\eta, \int_{y_K}^{y}\frac{\eta-y_K}{b^{\varepsilon}(\eta)}\,\mathrm{d}\eta,\ldots,\\ &\int_{x_K}^{x}\frac{(\xi-x_K)^{k-1}}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi, \int_{x_K}^{x}\frac{(\xi-x_K)^{k-2}}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\int_{y_K}^{y}\frac{1}{b^{\varepsilon}(\eta)}\,\mathrm{d}\eta,\ldots,\\ &\int_{x_K}^{x}\frac{1}{a^{\varepsilon}(\xi)}\,\mathrm{d}\xi\int_{y_K}^{y}\frac{(\eta-y_K)^{k-2}}{b^{\varepsilon}(\eta)}\,\mathrm{d}\eta, \int_{y_K}^{y}\frac{(\eta-y_K)^{k-1}}{b^{\varepsilon}(\eta)}\,\mathrm{d}\eta\Bigg\}\Bigg\}\end{aligned}$$

Here K is a 2-D cell with the barycenter at the point (x_K, y_K).

In the following lemma we will prove the approximation property for S_2^1.

Lemma 3.4

Let $u(x,y)$ be the exact solution of (22). There exists some $v(x,y)\in S_2^1$ such that for all cell K:

$$\|u-v\|_{L^2(K)}\leqslant C(\alpha,\beta)\|f\|_{L^2(K)}(\Delta_K)^2 \quad \forall(x,y)\in K \tag{24}$$

where $\Delta_K=\operatorname{diam}(K)$ and $C(\alpha,\beta)$ is a constant depending on α, β and independent of Δ_K and ε.

Proof

This lemma is a generalization of the theorems in [4] and so is the procedure of its proof. We define $\tilde{x}=\int_{x_K}^{x}(1/a^{\varepsilon}(\xi))\,\mathrm{d}\xi$, $\tilde{y}=\int_{y_K}^{y}(1/b^{\varepsilon}(\eta))\,\mathrm{d}\eta$, $\tilde{u}(\tilde{x},\tilde{y})=u(x,y)$, $\tilde{a}^{\varepsilon}(\tilde{x})=a^{\varepsilon}(x)$, $\tilde{b}^{\varepsilon}(\tilde{y})=b^{\varepsilon}(y)$, and $\tilde{f}(\tilde{x},\tilde{y})=f(x,y)$. Change the variables x and y in (22) to $\tilde{x}$ and $\tilde{y}$, we obtain a new elliptic PDE:

$$-\tilde{b}^{\varepsilon}\tilde{u}_{\tilde{x}\tilde{x}}-\tilde{a}^{\varepsilon}\tilde{u}_{\tilde{y}\tilde{y}}=\tilde{a}^{\varepsilon}\tilde{b}^{\varepsilon}\tilde{f}, \quad (\tilde{x},\tilde{y})\in\tilde{K} \tag{25}$$

From the Bernstein Theorem in [4], we have

$$\|\tilde{u}\|_{H^2(\tilde{K})}\leqslant C(\alpha,\beta)\|\tilde{a}^{\varepsilon}\tilde{b}^{\varepsilon}\tilde{f}\|_{L^2(\tilde{K})}\leqslant C(\alpha,\beta)\|f\|_{L^2(K)}$$

From the approximation theory of polynomial spaces, we can find a linear polynomial $\tilde{v}=c_0+c_1\tilde{x}+c_2\tilde{y}$ such that

$$\|\tilde{u}-\tilde{v}\|_{L^2(\tilde{K})}\leqslant C\|\tilde{u}\|_{H^2(\tilde{K})}(\Delta_{\tilde{K}})^2$$

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Now we denote $v(x, y) = \tilde{v}(\tilde{x}, \tilde{y}) = c_0 + c_1 \int_{x_K}^{x} (1/a^{\varepsilon}(\xi))\,\mathrm{d}\xi + c_2 \int_{y_K}^{y} (1/b^{\varepsilon}(\eta))\,\mathrm{d}\eta$. We also have $\Delta_{\tilde{K}} \leqslant C\Delta_K$. The approximation property can then be obtained:

$$\|u - v\|_{L^2(K)} \leqslant C(\alpha, \beta)\|\tilde{u} - \tilde{v}\|_{L^2(\tilde{K})} \leqslant C(\alpha, \beta)\|\tilde{u}\|_{H^2(\tilde{K})}(\Delta_{\tilde{K}})^2 \leqslant C(\alpha, \beta)\|f\|_{L^2(K)}(\Delta_K)^2$$

This completes the proof. □

Next, we will prove the L^2 approximation property for S_2^1.

Lemma 3.5
Let $u(x, y)$ be the exact solution of (22) and P_h be the L^2 projection operator into the space S_2^1. There exists a constant $C(\alpha, \beta)$ such that

$$\|u - P_h u\|_{L^2(\Omega)} \leqslant C(\alpha, \beta)\|f\|_{L^2(\Omega)}\Delta^2 \tag{26}$$

where $\Delta = \max_K \Delta_K$.

Proof
We choose the same v as that in Lemma 3.4. Squaring both sides of (24), we obtain

$$\|u - v\|_{L^2(K)}^2 \leqslant C(\alpha, \beta)\|f\|_{L^2(K)}^2(\Delta_K)^4 \quad \forall K$$

Therefore,

$$\|u - P_h u\|_{L^2(\Omega)}^2 \leqslant \|u - v\|_{L^2(\Omega)}^2 \leqslant C(\alpha, \beta)\sum_K \|f\|_{L^2(K)}^2(\Delta_K)^4 \leqslant C(\alpha, \beta)\|f\|_{L^2(\Omega)}^2\Delta^4$$

Taking square roots on both sides finishes the proof. □

4. BOUNDEDNESS AND STABILITY OF THE MULTI-SCALE DG OPERATOR, AND APPROXIMATION PROPERTIES

Here we denote the *primal form* $B_h(\cdot, \cdot)$ as the left-hand side of the *primal formulation* of the DG method. In this section, we discuss the boundedness and stability of B_h and the approximation properties of the space S^k with respect to some appropriate norm. For simplicity, we will only provide the proof for the Babuška–Zlámal DG method in 1-D following the proof in [8]. The proof can also be easily generalized to other DG methods and to 2-D along the lines of [8].

4.1. Boundedness

Lemma 4.1 (*Boundedness*)
There exists some constant C_b such that

$$B_h(w, v) \leqslant C_b |||w|||\,|||v||| \quad \forall w, v \in S^k \tag{27}$$

where the norm $|||\cdot|||$ is defined as

$$|||v|||^2 = |v|_{H^1(0,1)}^2 + |v|_*^2$$

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with the norm $|\cdot|_*$ defined as

$$|v|_*^2 = \sum_{j=0}^{N} (\Delta x)^{-2k-1} [v]_{j+1/2}^2$$

Proof
For the Babuška–Zlámal method, we have

$$B_h(w, v) = \sum_{j=1}^{N} \int_{I_j} a^{\varepsilon} w_x v_x \, \mathrm{d}x + \sum_{j=0}^{N} \left(\frac{a^{\varepsilon}(x_{j+1/2}) \eta_{j+1/2}}{\Delta x} [w]_{j+1/2} [v]_{j+1/2} \right)$$

From (12), we can easily obtain

$$\sum_{j=1}^{N} \int_{I_j} a^{\varepsilon} w_x v_x \, \mathrm{d}x \leqslant C |w|_{H^1(0,1)} |v|_{H^1(0,1)} \tag{28}$$

for some constant C. We also have

$$\sum_{j=0}^{N} \left(\frac{a^{\varepsilon}(x_{j+1/2}) \eta_{j+1/2}}{\Delta x} [w]_{j+1/2} [v]_{j+1/2} \right) \leqslant C_1 (\Delta x)^{2k} \left(\sup_j \eta_{j+1/2} \right) |w|_* |v|_* \tag{29}$$

for some constant C_1. Combining (28) and (29) and noticing that $(\Delta x)^{2k} (\sup_j \eta_{j+1/2}) = O(1)$ complete the proof. □

4.2. Stability

Lemma 4.2 (*Stability*)
There exists some constant C_s such that

$$B_h(v, v) \geqslant C_s |||v|||^2 \quad \forall v \in S^k \tag{30}$$

Proof
From (12), we can easily obtain

$$\sum_{j=1}^{N} \int_{I_j} a^{\varepsilon} v_x v_x \, \mathrm{d}x \geqslant C |v|_{H^1(0,1)}^2 \tag{31}$$

for some constant C. We also have

$$\sum_{j=0}^{N} \left(\frac{a^{\varepsilon}(x_{j+1/2}) \eta_{j+1/2}}{\Delta x} [v]_{j+1/2} [v]_{j+1/2} \right) \geqslant C_1 (\Delta x)^{2k} \left(\inf_j \eta_{j+1/2} \right) |v|_*^2 \tag{32}$$

for some constant C_1. The combination of (31) and (32) gives us (30). To make C_s independent of Δx, the lower bound of $\eta_{j+1/2}$ must be sufficiently large, i.e. $\inf_j \eta_{j+1/2} = \mathcal{O}((\Delta x)^{-2k})$. □

4.3. Approximation

Lemma 4.3 (*Approximation*)
Let $u(x)$ be the exact solution of (10). There exists some interpolant $u_I(x) \in S^k$ $(k \geqslant 1)$ such that

$$|||u - u_I||| \leqslant C_a (\Delta x)^k |f|_{H^{k-1}(0,1)} \tag{33}$$

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Proof
We choose u_I as the usual continuous interpolant, then the jumps of $u-u_I$ will vanish on the boundaries. We also have $|u-u_I|_{H^1(0,1)} \leqslant C(\Delta x)^k |f|_{H^{k-1}(0,1)}$, proven in [5]. Then

$$|||u-u_I||| = |u-u_I|_{H^1(0,1)} \leqslant C(\Delta x)^k |f|_{H^{k-1}(0,1)}$$

The proof is completed. □

5. ERROR ESTIMATES

We now prove an error estimate for the DG method based on the multi-scale approximation spaces by using the properties of consistency, boundedness, stability, and approximation discussed previously. Again, only the results for the Babuška–Zlámal DG method are provided.

Lemma 5.1 (*Error estimates*)
Let $u(x)$ be the exact solution of (10) and u_h be the numerical solution computed by the multi-scale Babuška–Zlámal DG method. There exists some constant C independent of ε such that

$$\|u-u_h\|_{L^2(0,1)} \leqslant C(\Delta x)^{k+1} |f|_{H^{k-1}(0,1)} \tag{34}$$

Proof
The proof here mainly follows the lines in [8]. The Babuška–Zlámal DG method is not consistent. Instead of satisfying the consistency condition, it satisfies

$$B_h(u,v) = \int_0^1 f v \, \mathrm{d}x + \sum_{j=0}^{N} a^{\varepsilon}(x_{j+1/2}) \{u_x\}_{j+1/2} [v]_{j+1/2} \quad \forall v|_{I_j} \in H^2(I_j) \ \forall j \tag{35}$$

where u is the exact solution of (10) and $\{w\} = \frac{1}{2}(w^- + w^+)$.

This method is neither adjoint consistent. We also have

$$B_h(v,\psi) = \int_0^1 v g \, \mathrm{d}x + \sum_{j=0}^{N} a^{\varepsilon}(x_{j+1/2}) [v]_{j+1/2} \{\psi_x\}_{j+1/2} \quad \forall v|_{I_j} \in H^2(I_j) \ \forall j$$

where ψ is the exact solution of the adjoint problem of (10), which is

$$-(a^{\varepsilon}(x)\psi_x)_x = g(x), \quad 0 \leqslant x \leqslant 1 \tag{36}$$

with the boundary condition

$$\psi(0) = \psi(1) = 0 \tag{37}$$

From the stability of the Babuška–Zlámal method, we obtain

$$C_s |||u_I - u_h|||^2 \leqslant B_h(u_I - u, u_I - u_h) + B_h(u - u_h, u_I - u_h) =: T_1 + T_2 \tag{38}$$

Since u_I is the continuous interpolant of u in S^k, we can use the continuity of $u-u_I$ to estimate T_1:

$$T_1 \leqslant C |||u_I - u||| \, |||u_I - u_h||| \leqslant C(\Delta x)^k |||u_I - u_h||| \, |f|_{H^{k-1}(0,1)} \tag{39}$$

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We also have

$$T_2 = \sum_{j=0}^{N} a^{\varepsilon}(x_{j+1/2})\{u_x\}_{j+1/2}[u_I - u_h]_{j+1/2}$$

$$\leqslant C(\Delta x)^k |||u_I - u_h|||\, \|u\|_{H^2(0,1)} \leqslant C(\Delta x)^k |||u_I - u_h|||\, \|f\|_{L^2(0,1)} \tag{40}$$

where the second inequality comes from the regularity of the elliptic PDE (10) and the first inequality is estimated by using the auxiliary inequality

$$\sum_{j=0}^{N} a^{\varepsilon}(x_{j+1/2})\{u_x\}_{j+1/2}[v]_{j+1/2} = \sum_{j=0}^{N} a^{\varepsilon}(x_{j+1/2})(\Delta x)^{k+1/2}\{u_x\}_{j+1/2}[v]_{j+1/2}(\Delta x)^{-k-1/2}$$

$$\leqslant C|||v|||\left(\sum_{j=0}^{N}(\Delta x)^{2k+1}\{u_x\}_{j+1/2}^2\right)^{1/2}$$

$$\leqslant C(\Delta x)^k |||v|||\, \|u\|_{H^2(0,1)} \tag{41}$$

with the last step following from the trace inequality which is proven in [2].

Substituting (39) and (40) back into (38), we get

$$|||u_I - u_h||| \leqslant C(\Delta x)^k \|f\|_{H^{k-1}(0,1)}$$

and

$$|||u - u_h||| \leqslant C(\Delta x)^k \|f\|_{H^{k-1}(0,1)}$$

by the triangle inequality.

For the L^2-error estimate, we use (36) with $g = u - u_h$ to obtain

$$\|u - u_h\|_{L^2(0,1)}^2 = B_h(u - u_h, \psi) - \sum_{j=0}^{N} a^{\varepsilon}(x_{j+1/2})\{\psi_x\}_{j+1/2}[u - u_h]_{j+1/2} =: T_3 + T_4 \tag{42}$$

Let ψ_I be the continuous interpolant of ψ in S^k, then $B_h(u, \psi_I) = \int_0^1 f\psi_I\,\mathrm{d}x$. Therefore,

$$T_3 = B_h(u - u_h, \psi - \psi_I) \leqslant C|||u - u_h|||\, |||\psi - \psi_I||| \leqslant C\Delta x |||u - u_h|||\, \|u - u_h\|_{L^2(0,1)} \tag{43}$$

where for the last inequality we have used Lemma 4.3 with u replaced by ψ.

The term T_4 can be estimated by using the auxiliary inequality (41):

$$T_4 \leqslant C(\Delta x)^k |||u - u_h|||\, \|\psi\|_{H^2(0,1)} \leqslant C(\Delta x)^k |||u - u_h|||\, \|u - u_h\|_{L^2(0,1)} \tag{44}$$

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Substituting (43) and (44) back into (42), we get

$$\|u-u_h\|_{L^2(0,1)}\leqslant C(\Delta x)^{k+1}\|f\|_{H^{k-1}(0,1)}$$

which finishes the proof. □

6. A NUMERICAL EXAMPLE

In this section, we present a numerical example of using the Babuška–Zlámal DG method based on the proposed multi-scale approximation spaces for solving the elliptic multi-scale problem (10) with

$$f=x,\quad a=\frac{1}{2+x+\sin(2\pi x/\varepsilon)} \tag{45}$$

with $\varepsilon=0.1$ and 0.01, respectively. We take $\eta_{j+1/2}=\Delta x^{-2k}$ in the computation. The numerical results are shown in Table II. We can clearly observe that the solutions are numerically well resolved and the expected $(k+1)$th order of accuracy is achieved starting from Δx larger than ε, without any resonance effects when the mesh size Δx changes from larger than ε to smaller than ε. Our numerical implementation does show an amplification of the round-off errors to around the level at 10^{-7} in double precision, probably due to the large condition number of the mass matrix. We will explore more accurate implementation of the method in the future. For comparison,

Table II. L^2-errors for the Babuška–Zlámal DG method based on multi-scale approximation spaces and regular piecewise polynomials. N uniform cells.

	$\varepsilon=0.1$				$\varepsilon=0.01$			
	S^1 space		P^1 space		S^1 space		P^1 space	
N	L^2-error	Order	L^2-error	Order	L^2-error	Order	L^2-error	Order
10	9.36E − 04		8.37E − 03		7.06E − 04		9.46E − 03	
20	2.37E − 04	1.98	1.87E − 03	2.16	1.72E − 04	2.00	9.82E − 03	−0.05
40	6.12E − 05	1.95	1.91E − 03	−0.03	4.40E − 05	1.97	9.71E − 03	0.02
80	1.57E − 05	1.96	5.03E − 04	1.92	1.13E − 05	1.97	9.72E − 03	0.00
160	3.96E − 06	1.99	1.28E − 04	1.97	2.59E − 06	2.06	7.75E − 03	0.33
320	9.92E − 07	2.00	3.21E − 05	2.00	7.13E − 07	1.93	2.80E − 03	1.47
640	2.48E − 07	2.00	8.02E − 06	2.00	1.85E − 07	1.94	7.74E − 04	1.86
1280	1.17E − 07	1.08	1.87E − 07	2.10	1.19E − 07	0.45	1.99E − 04	1.96
	S^2 space		P^2 space		S^2 space		P^2 space	
N	L^2-error	Order	L^2-error	Order	L^2-error	Order	L^2-error	Order
5	3.20E − 04		8.38E − 03		3.20E − 04		9.73E − 03	
10	2.08E − 05	3.94	3.94E − 03	1.09	2.08E − 05	3.94	9.96E − 03	−0.03
20	2.00E − 06	3.38	1.91E − 03	1.04	1.98E − 06	3.39	9.80E − 03	0.02
40	2.35E − 07	3.09	9.03E − 05	4.40	2.33E − 07	3.09	9.73E − 03	0.01
80	3.76E − 08	2.64	7.40E − 06	3.61	7.27E − 08	1.68	9.17E − 03	0.09

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we also list in Table II the results using the Babuška–Zlámal DG method based on the DG space of piecewise polynomials. We notice that we do not observe the expected order of convergence until the mesh is refined enough relative to ε, which is consistent with the error estimates for such DG method based on regular piecewise polynomials.

7. CONCLUDING REMARKS

DG methods based on multi-scale approximation spaces for solving a class of second-order elliptic PDEs with multi-scale solutions are studied in this paper. The basis functions of the multi-scale approximation spaces are constructed from the differential operators to capture the micro-scale structure information of solutions so that the solutions of multi-scale PDEs can be well approximated. The resulting method can be shown to achieve the designed convergence rate, based on the regularity of the right-hand side of the PDE (1) which is independent of ε, not on the regularity of the actual solution of this PDE which depends on ε. Both theoretical proofs and numerical experiments show that, compared with the usual piecewise polynomial spaces, more accurate results are obtained for coarse meshes. No resonance effects appear, that is, the rate of convergence is independent of ε. Even though the theory and numerical results are shown for the Babuška–Zlámal DG method only, the proposed multi-scale DG methodology is not restricted to any specific DG formulation.

From a practical point of view, we have only considered the 1-D problem and a special class of 2-D problems for which the multi-scale basis functions can be explicitly constructed. This renders the DG scheme very efficient, with a cost comparable with that of the regular DG scheme with polynomial basis functions. The method also works for more general multi-dimensional problems; however, the multi-scale basis functions may not be available explicitly and must also be computed numerically [7], which would increase the computational cost of the multi-scale DG method tremendously.

In future work we will generalize both the multi-scale DG method and its analysis to broader classes of multi-dimensional problems with higher order of accuracy. We will also explore applications of the multi-scale DG method in fluid dynamics and semi-conductor device simulations.

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